Units Coordination across Whole Numbers, Fractions, and Early Algebra

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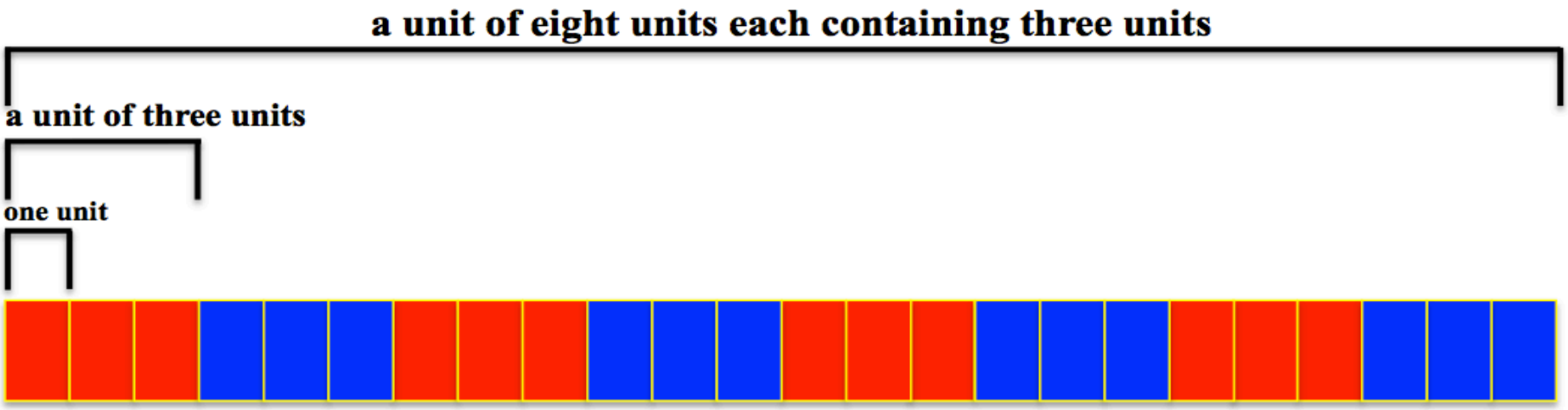
Plan

- What is a units coordination?
- How do students' stages of units coordination influence students' mathematical thinking?
- How can we understand and assess these different stages?

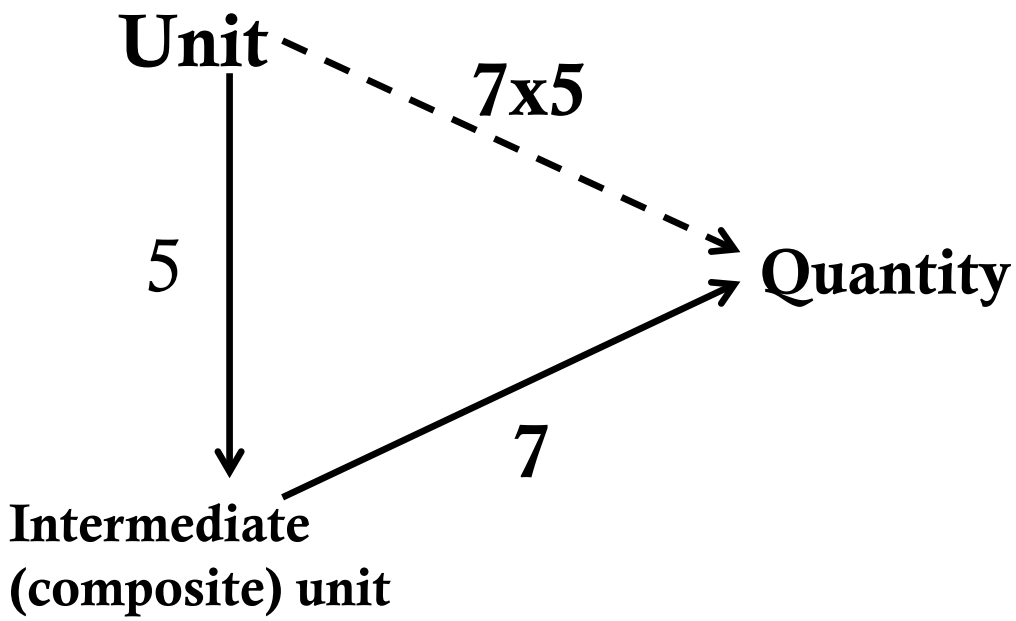
Units Coordination



Stages of Units Coordinating

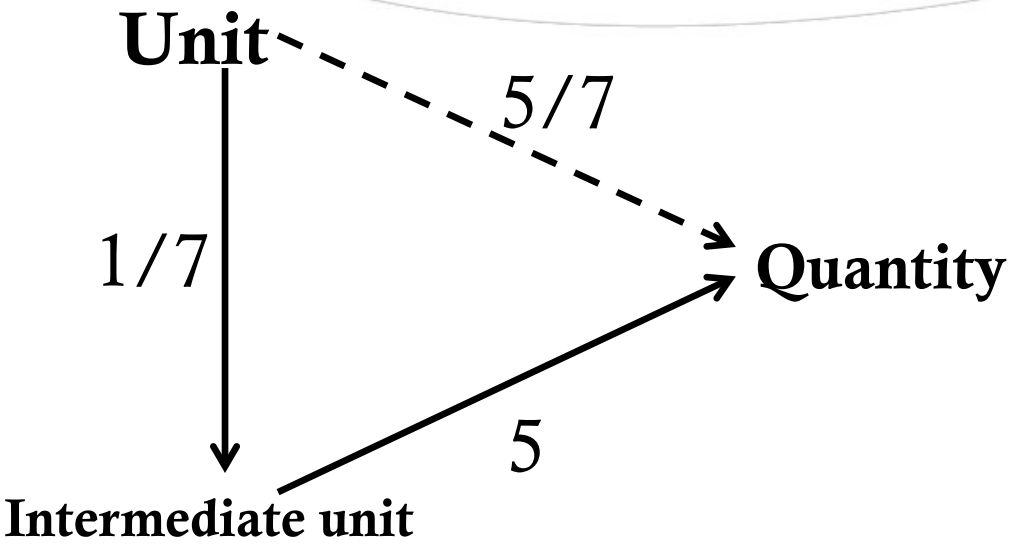
- Stage 1: Units of units in activity
- Stage 2: Units of units prior to activity
- Stage 3: Units of units of units prior to activity

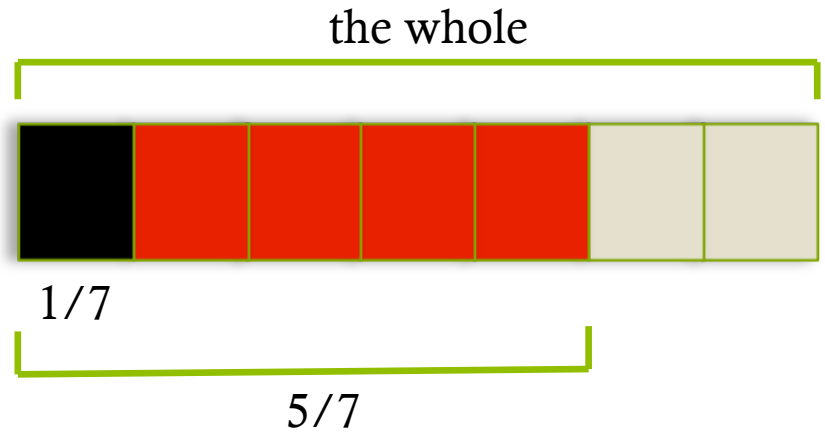
Whole Number Multiplication



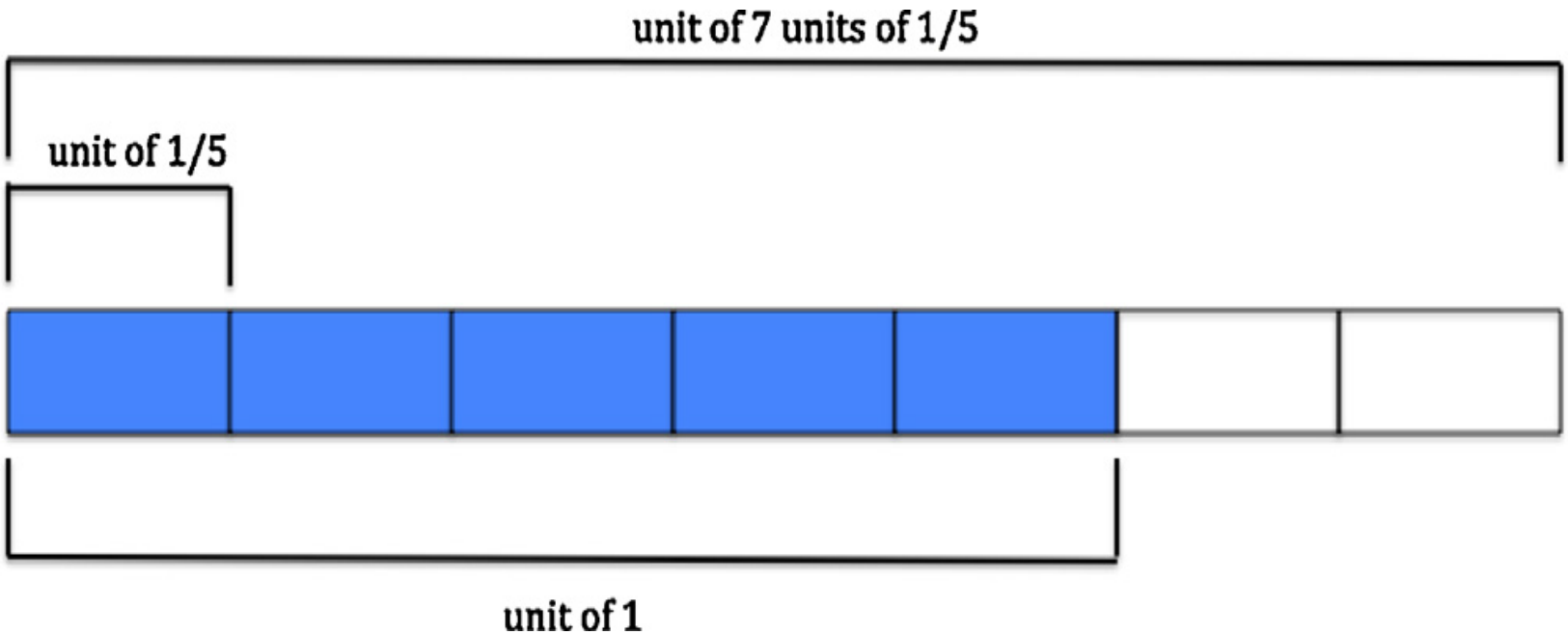
35 as a Unit of Seven Units of Five 1s

Unit Fractions as Measures



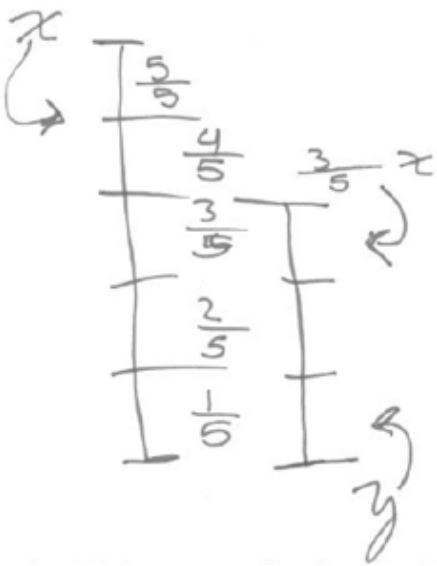


The Iterative Fraction Scheme



Reciprocal Reasoning

The height of a sunflower is 3/5 the height of a fern.
Draw a picture to show the two heights.
Write equations to represent the relationship between the heights.



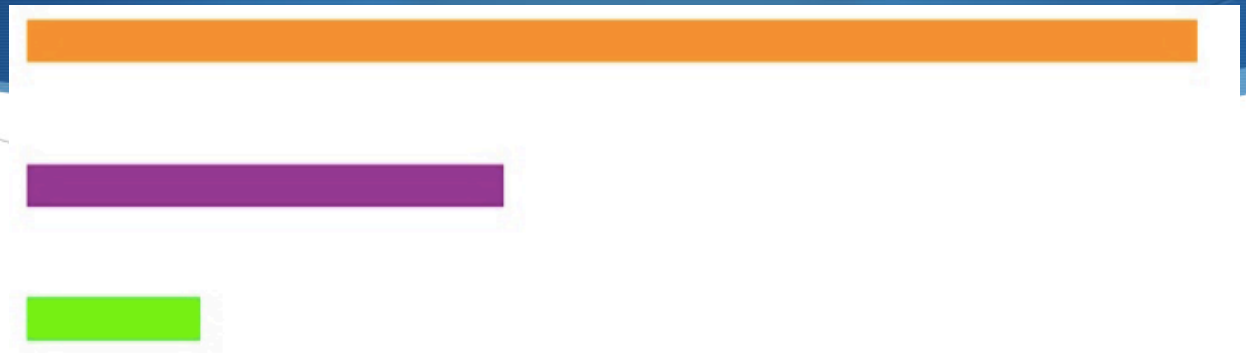
x = height of F in cm
y = height of SF in cm

y = (3/5)x

x = (5/3)y because each 1/5 of x is equal to 1/3 of y.

Bars Tasks

Bars Tasks: Chapter 3



Use the following information to answer questions about the bars shown above:

4. **Pretend** that the **Medium Purple Bar** fits into the **Long Orange Bar** ***exactly*** **2** times.

   Pretend that the **Small Green Bar** fits into the **Medium Purple Bar** ***exactly*** **6** times.

   Use this information to figure out how many times the **Small Green Bar** would fit into the **Long Orange Bar**?

   answer: ☐

   Use the space below to **draw a picture and explain** your answer.

Cody: a Stage 1 student

Units Coordinating Rubric

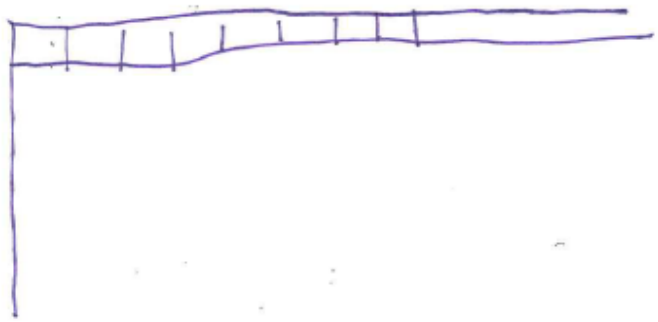
	Students' Unit Structures	Student Reasoning on Task 4	Written Indicators of Reasoning
Stage 1	Students can take one level of units as given, and may coordinate two levels of units in activity.	Students rely upon the appearance of the bars without using given relations.	• Students rely upon the appearance of the bars rather than using the given relations (e.g., partitioning/segmenting the given bars). • Students add or subtract the numbers given in the relations. • Students do not respond, or otherwise indicate they do not know.

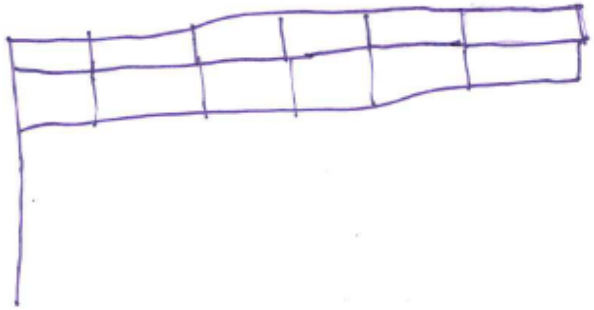
The Crate Problem

- There are 4 cans of juice in a package and 8 packages in a box. A crate contains 6 boxes.
- How many cans of juice are in a crate, and can you draw a picture to show how you know?
- *Follow-up:* Sometimes people do 6 x 8 in solving this problem. Does that make sense? What would 6 x 8 mean?

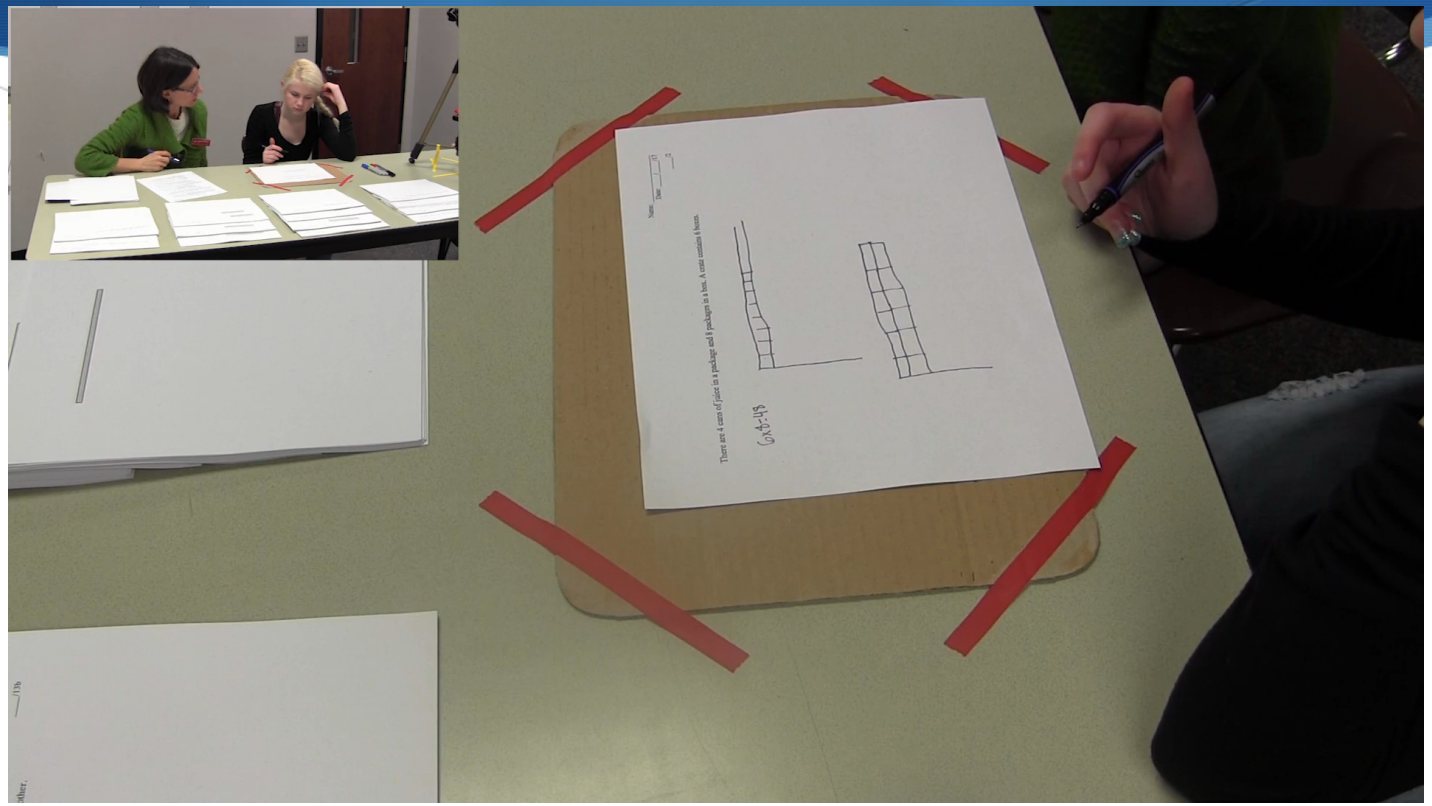
Alyssa: Crate Problem

6x8=48

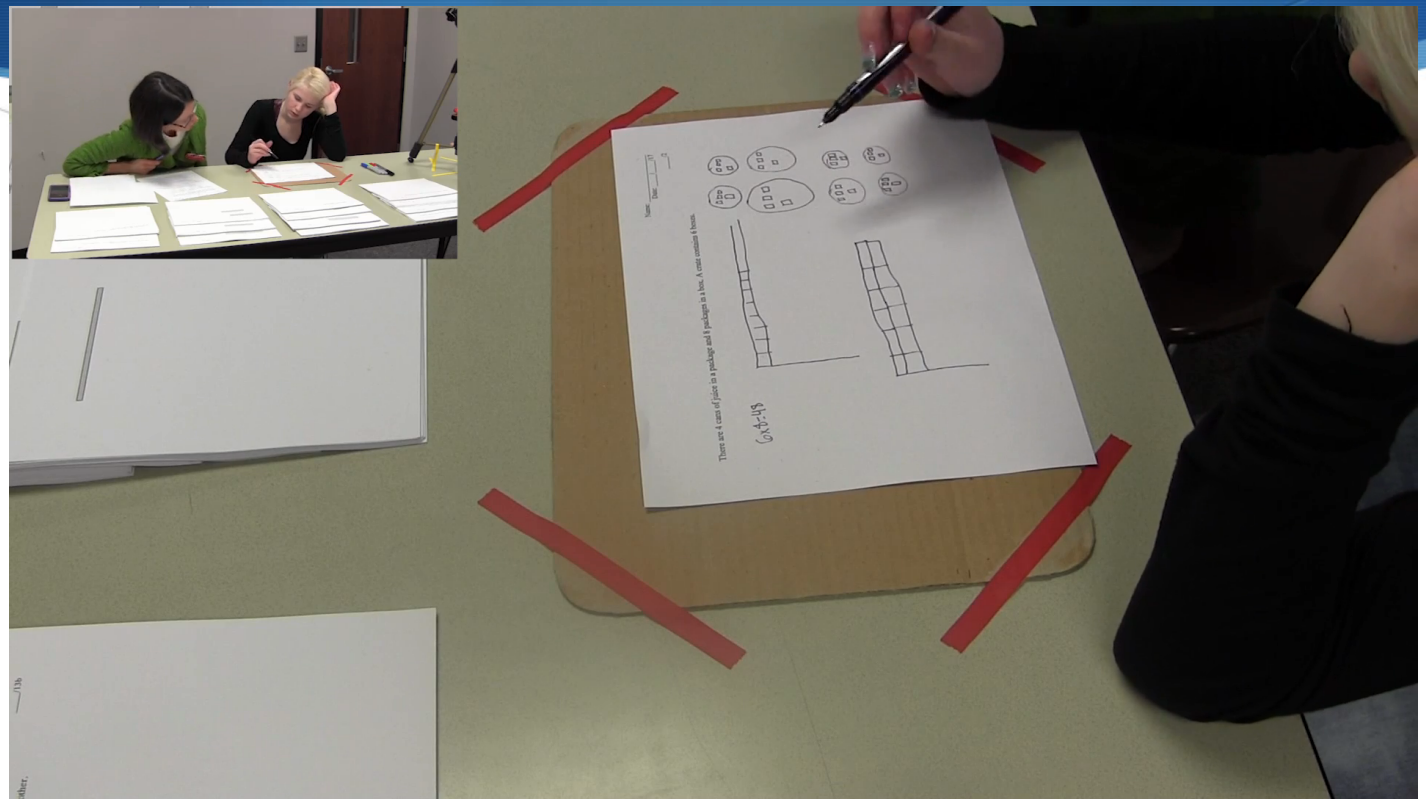




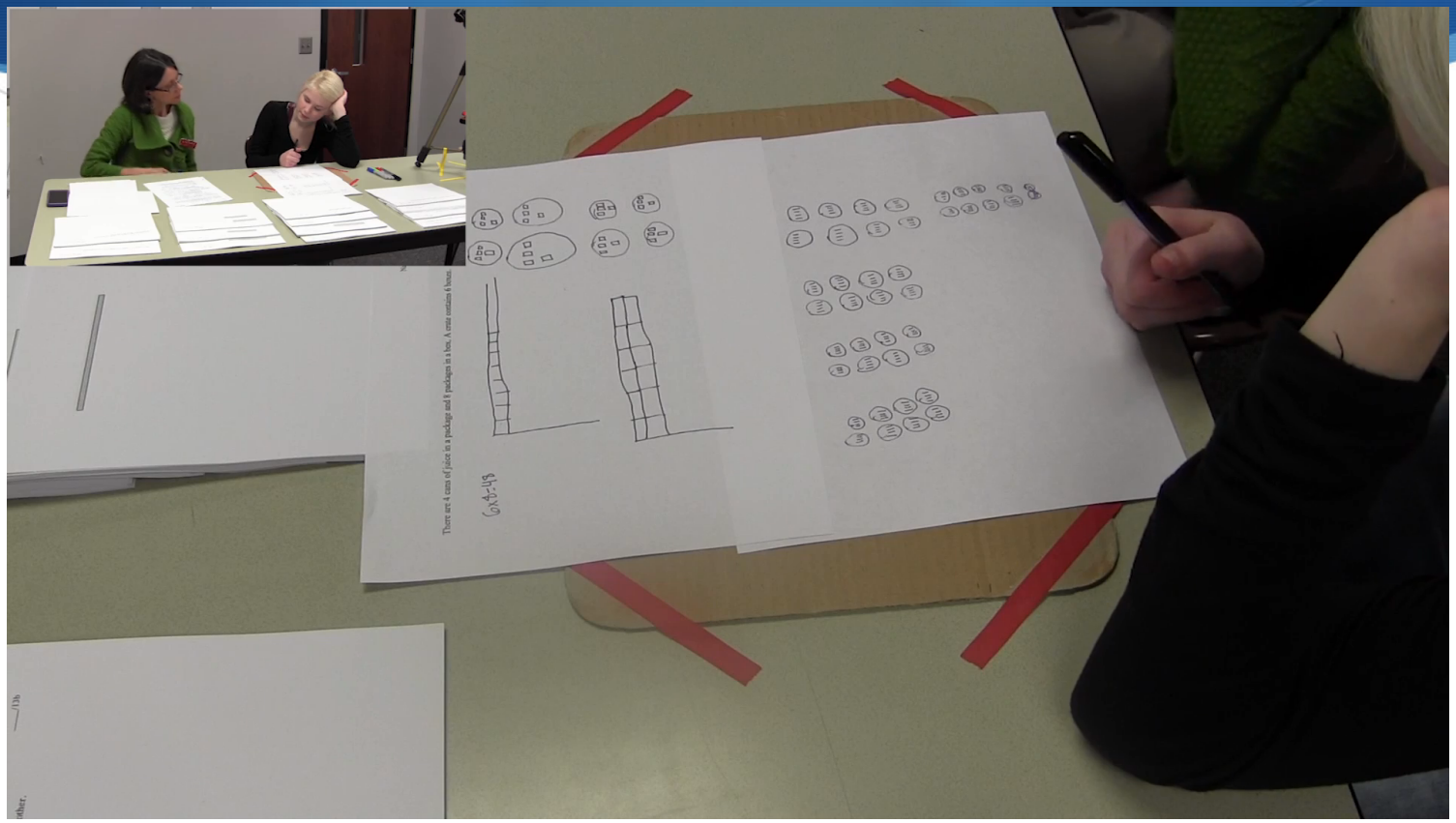
Alyssa: new picture



Alyssa: continuing toward a crate?



Alyssa: reflection



Observations about Alyssa

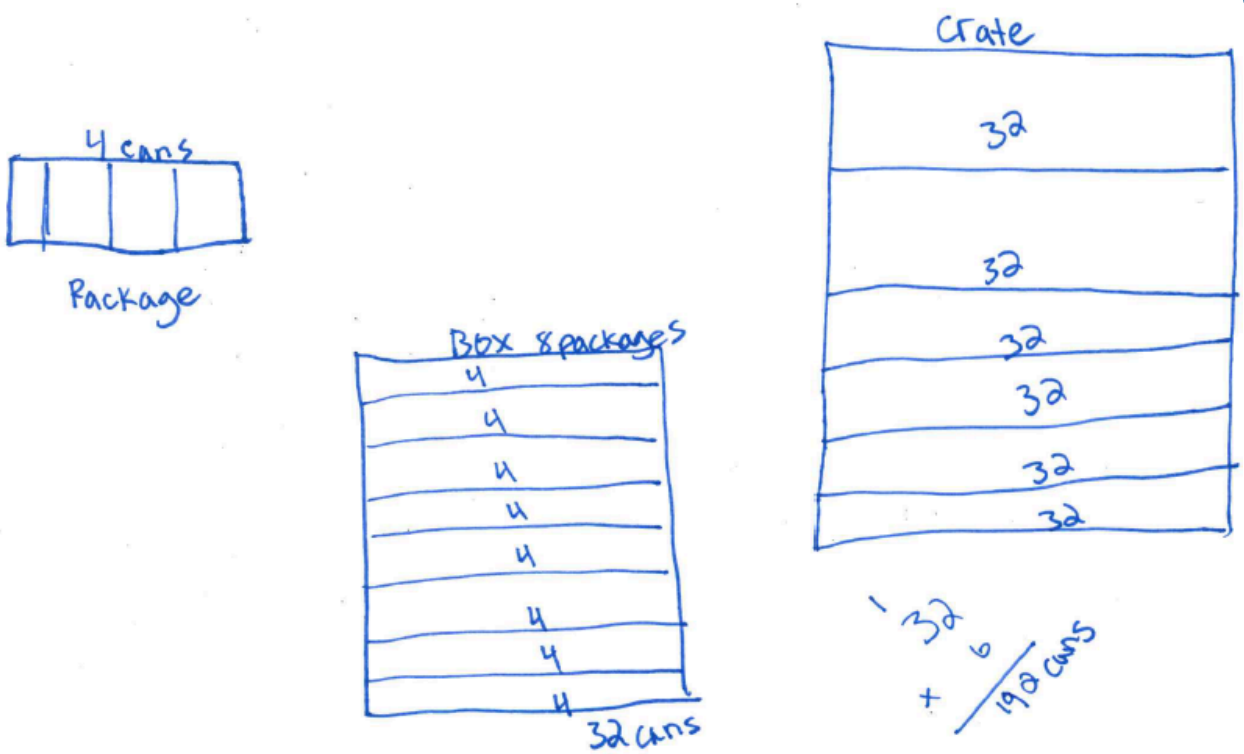
- She takes 4 as a unit and iterates it toward a goal, but the result is not a unit that she can operate on further conceptually.
- With support she drew eight 4s a total of six times—that is more than some stage 1 students demonstrate!
  - However, what she drew did not represent a crate to her, so conceptually she did not see the result as 6 boxes, each containing 8 packages with 4 cans in a package.
- What was the result to her? Probably a sea of packages. Even the box, which she identified seemed, to be ephemeral.

Vivian: a Stage 2 student

Units Coordinating Rubric

Stage 2	Students can take two levels of units (a composite unit) as given, and may coordinate three levels of units in activity.	Students use the second given relation to form a composite unit that they can iterate through activity, by the number in the first given relation.	• Students coordinate relations appropriately and with a drawing illustrating size relations, but writing indicates the drawing was the solution method (e.g., solution appears below the drawing, or erasures/corrections are present in the drawing). • Student explanations and drawings appropriately refer to multiple two-level relations, but not a single three-level relation. • Student responses indicate use of multiplication without justification or illustration (possibly with a multiplication error).

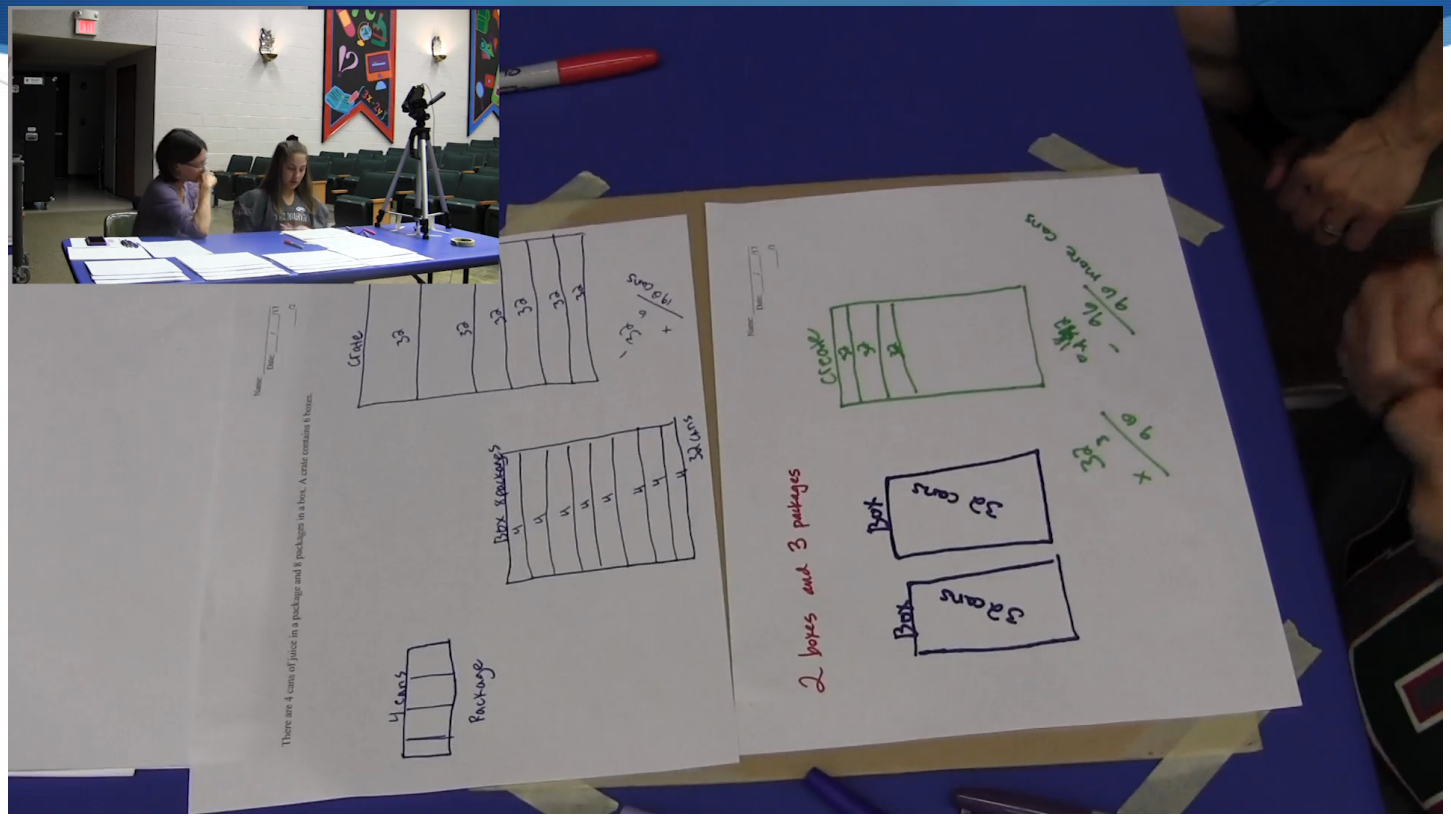
Joanna: Crate Problem



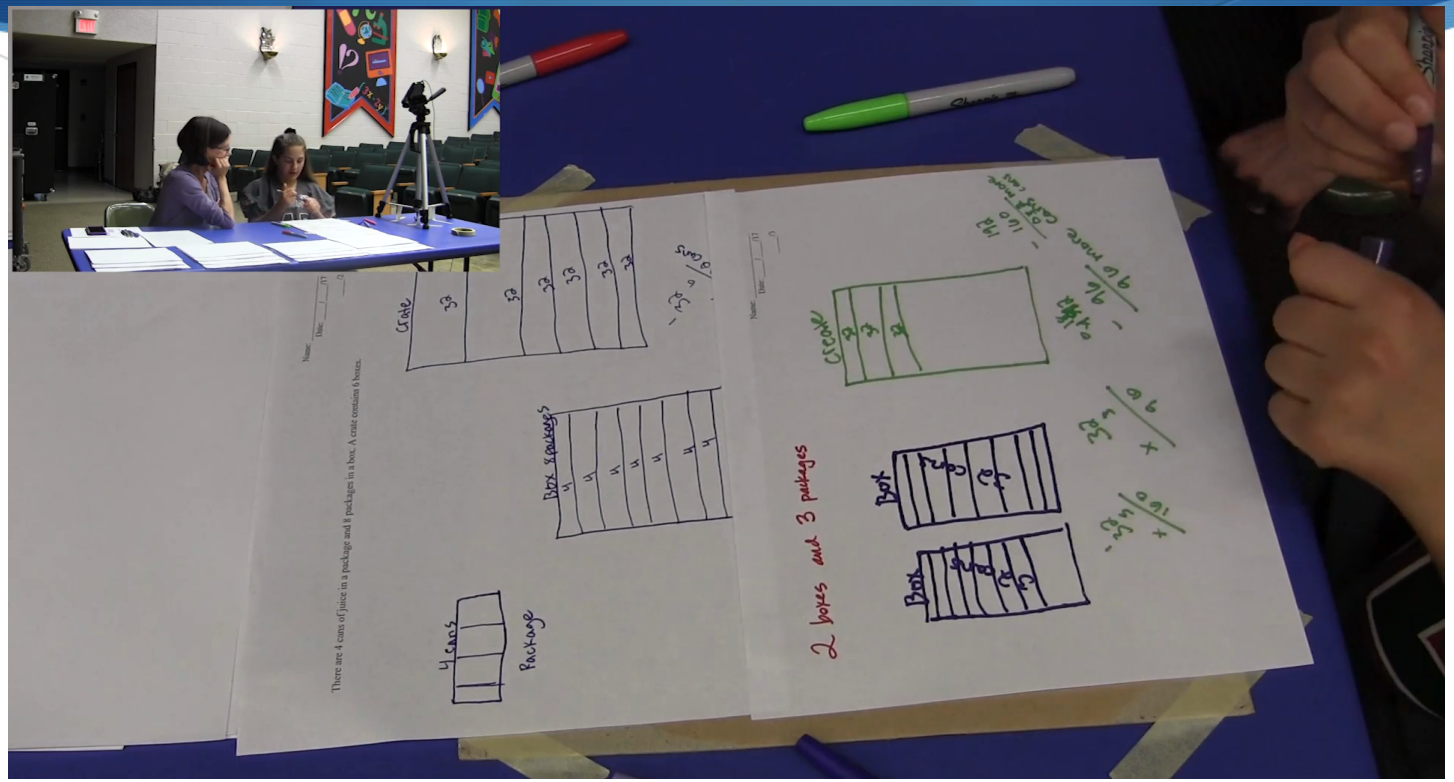
Partially Filled Crate Problem

- *Same situation:* There are 4 cans of juice in a package and 8 packages in a box. A crate contains 6 boxes.
- A worker has packed up 2 boxes and 3 packages. How many more cans does she need to pack up the whole crate?
- *Follow-up:* How will those cans be organized into packages and boxes?

Joanna: Partially Filled Crate



Joanna: Are there packages in the crate?



Observations about Joanna

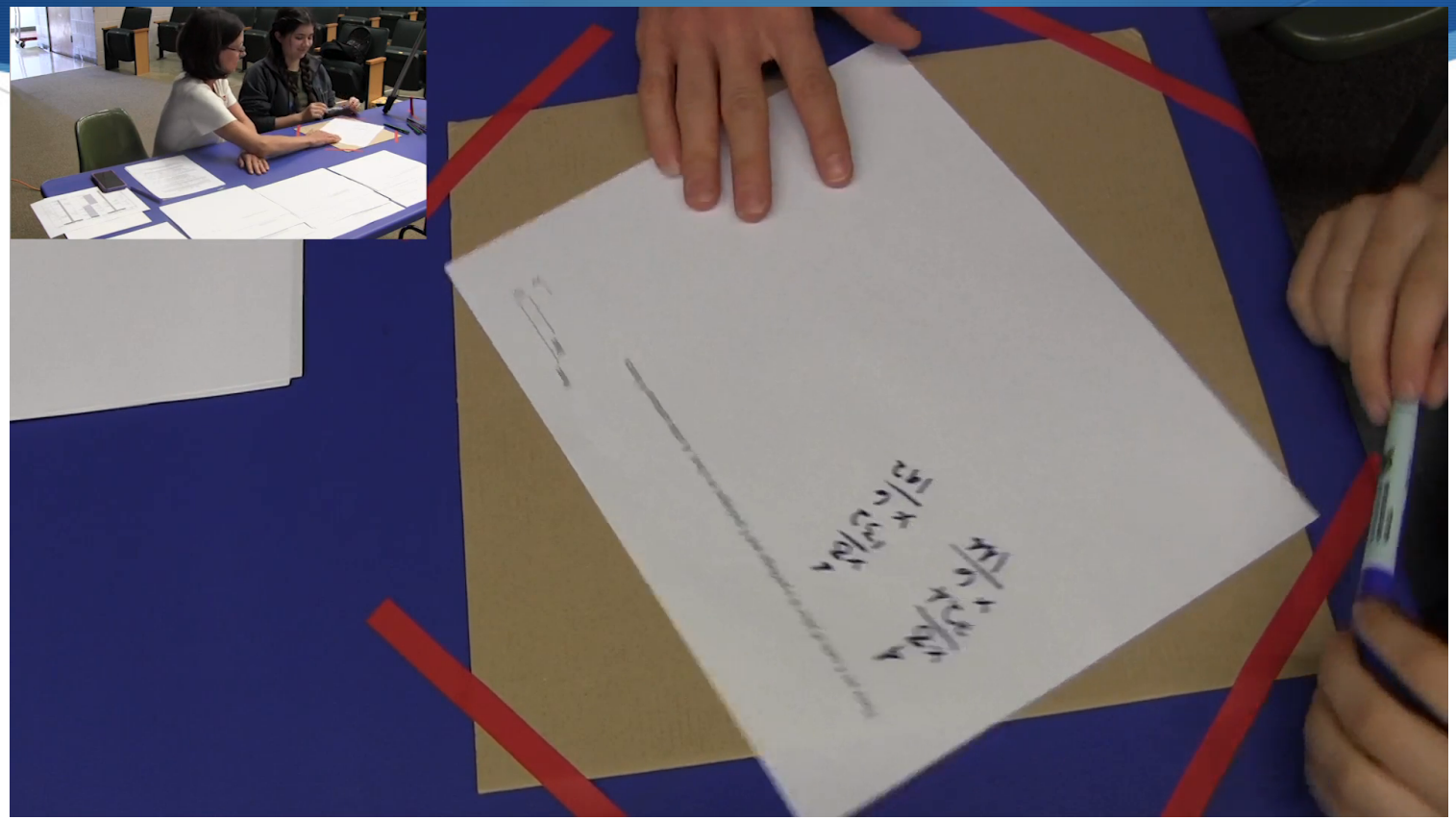
- In contrast with Alyssa, Joanna took eight 4s as a unit to operate with further.
- But in operating further, the 32 is not a unit of eight 4s: It is not a unit of units of units. Instead, it is a unit of 32 (ones), a unit of units.
  - However, Joanna builds the crate with this and gets a correct answer.
- So in the PFC problem, the packages and boxes become conflated because a crate is made from 6 32s, and the box is made from 8 4s.
- In addition, she believes that there are packages in the crate and does use a units coordination, six 8s, to get the number of packages.
- But that is not coordinated with the eight 4s and the 32.

Jimmie: a Stage 3 student

Units Coordinating Rubric

Stage 3	Students can take three levels of units (a composite unit of composite units) as given, and can thus flexibly switch between two and three-level structures without reliance on figurative material.	Students take the first given relation as a composite unit that they mentally distribute across the units given in the second relation, thus justifying the use of multiplication.	• Student drawings are used to justify or illustrate appropriate solutions rather than to produce them (e.g., drawing is integrated with or appears below an explanation). • Student explanations and drawings refer to a single three-level relation, with appropriate size relations.

Isabel: Crate Problem



Isabel: Partially Filled Crate

2 boxes and 3 packages

p.2 /3

1 box = 32 cans

1 pack. = 4 cans

$32 \times 2 = 64$

$4 \times 3 = 12$

$64 + 12 = 76$ cans

$192 - 76 = 116$ cans

29 packs

$116 \div 4 = 29$

-8

36

$298 \div 8 = 3.5$ R5

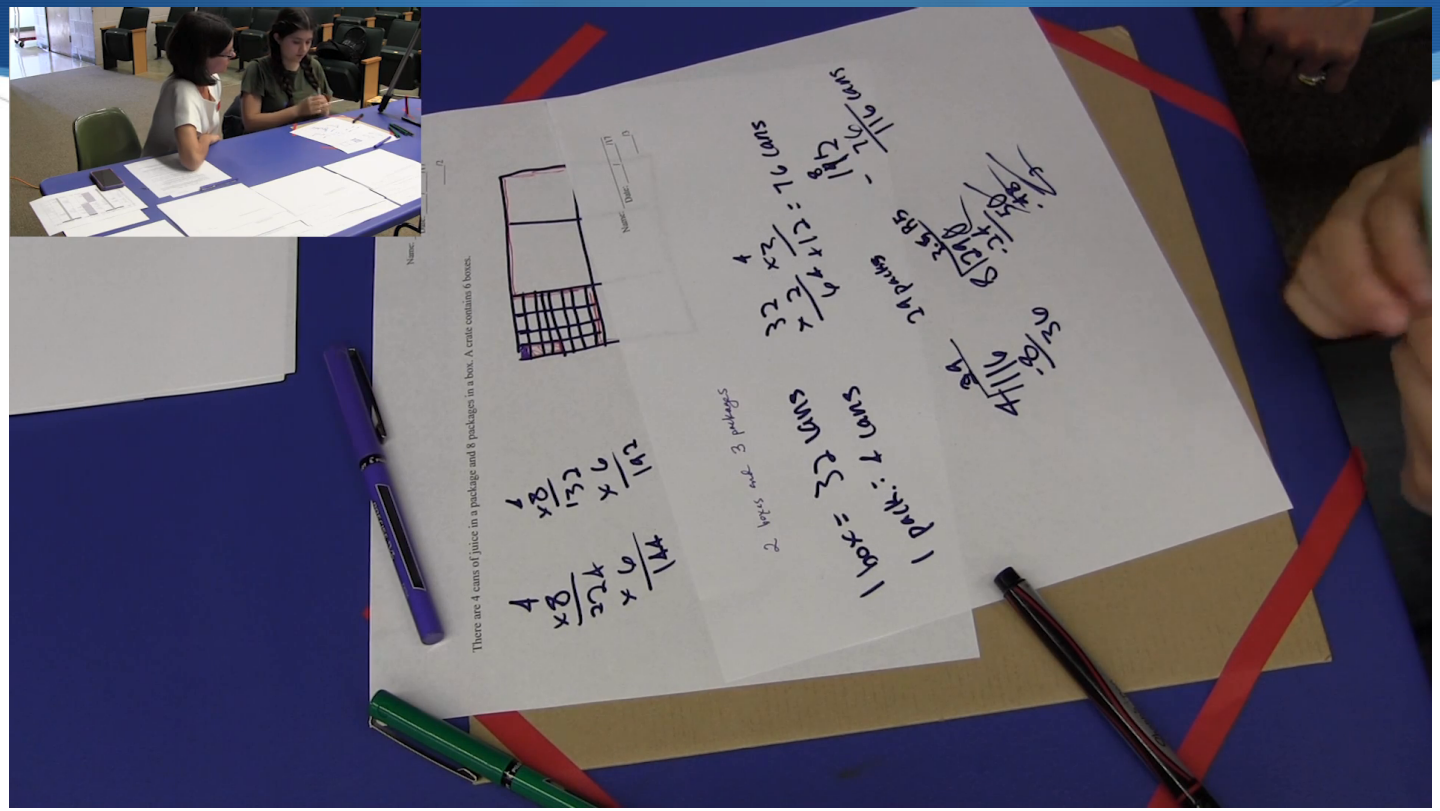
-24

50

-48

2

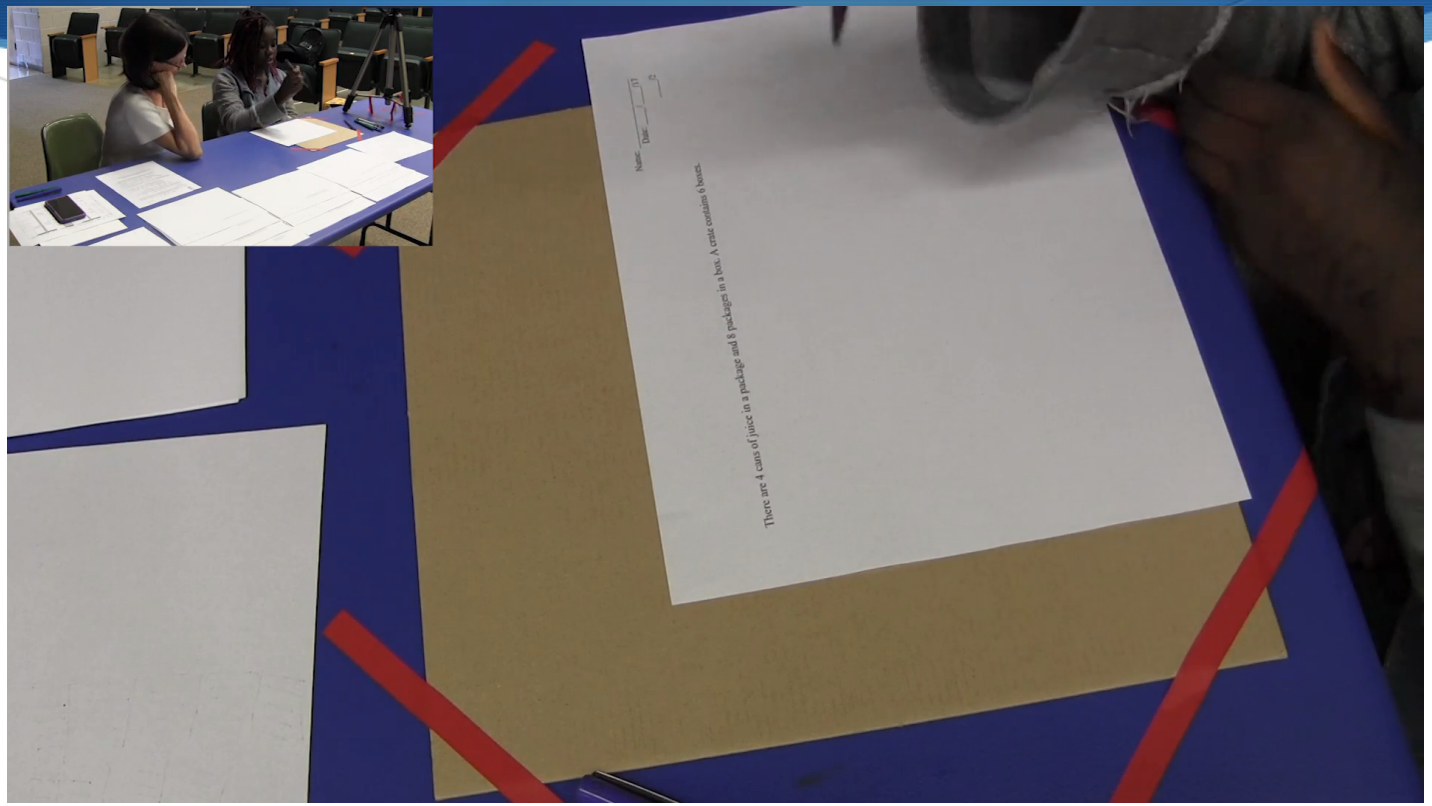
Isabel: What does 3 R 5 mean?



Observations about Isabel

- Her picture of the crate shows clear embeddedness of units within units within units.
- On the PFC problem, she works with multiple, embedded units (cans, packages, and boxes):
  - Repeated division is meaningful.
  - She interprets her response (3 boxes, 5 packages) in relation to the given number of boxes and packages.
- When conflations happen, she corrects those fairly smoothly and quickly (with some questioning support).

Whitney: a Stage ___ student



Observations about Whitney

Concluding Remarks

- What behavioral indicators did you notice for students operating at Stages 1, 2, and 3?
- At each stage, how might we productively engage in fractions tasks?
- How might we engage students in units coordination activity that promotes development toward the next stage?

Final Thoughts

- Units coordination is a tool for orientation and interaction!
  - Not stratification
- Design to meet students where they are
- Understand constraints
- Imagine/test possibilities

Thank you!

- **IDR^{2}eAM Project**: Investigating Differentiated Instruction and Relationships between Rational Number Knowledge and Algebraic Reasoning in Middle School
  - **www.indiana.edu/~idream**

- **Many thanks to the IDR^{2}eAM project team!** Fetiye Aydeniz, Mark Creager, Anna Dinndorf, Rebecca Borowski, Ayfer Eker, Sharon Hoffman, Robin Jones, Mi Yeon Lee, Rob Matyska, Pai Suksak.